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Strong convergence theorems for a semigroup of asymptotically nonexpansive mappings

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ABSTRACT

Let K be a nonempty closed convex subset of a real Banach space E. Let $\mathcal{T} := \{T(t) : t \geq 0\}$ be a strongly continuous semigroup of asymptotically nonexpansive mappings from K into K with a sequence $\{L_t\} \subset [1, \infty)$. Suppose $F(\mathcal{T}) \neq \emptyset$. Then, for a given $u \in K$ there exists a sequence $\{u_n\} \subset K$ such that $u_n = (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds + \alpha_n u$, for $n \in \mathbb{N}$, where $t_n \in R^+$, $\{\alpha_n\} \subset (0, 1)$ and $\{L_t\}$ satisfy certain conditions. Suppose, in addition, that E is reflexive strictly convex with a Gâteaux differentiable norm. Then, the sequence $\{u_n\}$ converges strongly to a point of $F(\mathcal{T})$. Furthermore, an *explicit* sequence $\{x_n\}$ which converges strongly to a fixed point of $\mathcal{T}$ is proved.

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1. Introduction

Let K be a closed convex subset of a Hilbert space H. One parameter family $\mathcal{T} := \{T(t) : t \geq 0\}$, denotes the set of nonnegative real numbers, is said to be *strongly continuous semigroup of Lipschitzian mappings* from K into K if the following conditions are satisfied:

(1) $T(0)x = x$ for all $x \in K$;
(2) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;
(3) for each $t > 0$, there exists a bounded measurable function $L_t : (0, \infty) \to [0, \infty)$ such that $\|T(t)x - T(t)y\| \leq L_t\|x - y\|$, $x, y \in K$;
(4) for each $x \in K$, the mapping $T(.)x$ from $\mathbb{R}^+ = [0, \infty]$ into K is continuous.

A strongly continuous semigroup of Lipschitzian mappings $\mathcal{T}$ is called *strongly continuous semigroup of nonexpansive mappings* if $L_t = 1$ for all $t > 0$, and *strongly continuous semigroup of asymptotically nonexpansive* if $\limsup_{t\to\infty} L_t \leq 1$. Note that for asymptotically nonexpansive semigroup $\mathcal{T}$, we can always assume that the Lipschitzian constant $\{L_t\}_{t>0}$ are such that $L_t \geq 1$ for each $t > 0$, L_t is non-increasing in t, and $\lim_{t\to\infty} L_t = 1$; otherwise we replace L_t, for each $t > 0$, with $\bar{L}_t := \max\{\sup_{s\geq t} L_s, 1\}$. $\mathcal{T}$ is said to have a fixed point if there exists $x_0 \in K$ such that $T(t)x_0 = x_0$, for all $t \geq 0$. We denote by $F(\mathcal{T})$, the set of fixed points of $\mathcal{T}$, i.e., $F(\mathcal{T}) := \cap_{t\geq 0} F(T(t))$.

A continuous operator of semigroup $\mathcal{T} := \{T(t) : t \geq 0\}$, is said to be *uniformly asymptotically regular* on K if for all $h \geq 0$ and any bounded subset C of K, $\lim_{t\to\infty}\sup_{x\in C}\|T(h)T(t)x - T(t)x\| = 0$.

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